CLAMFIM Bologna

Modelli 1 @ Clamfim

Ottimizzazione libera e vincolata

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Avviso

Eccezionalmente la lezione di mercoledì 24 settembre inizierà alle ore 8 a causa di impegni istituzionali inderogabili

Positive homogeneous functions

A function $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is positive homogeneous of degree k if for any $\boldsymbol{x} \in: \mathbb{R}^n \setminus \{0\}$

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Euler's homogeneous function theorem states that if $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is continuously differentiable, then f is positive homogeneous of degree k if and only if

$$\boldsymbol{x} \cdot \nabla f(\boldsymbol{x}) = k f(\boldsymbol{x})$$

Given

$$f(x,y)=\frac{\sqrt{x^2+y^2}}{xy}$$

establish if it is positive homogeneous and, in case, establish the degree of homogeneity. Then verify thesis of Euler theorem

Critical points

Definition. Let V be an open set in $\mathbb{R}^n$, let $\boldsymbol{a} \in V$ and suppose that $f : V \to \mathbb{R}$.

(i) $f(\boldsymbol{a})$ is called a *local minimum* of f if and only if there is an $r > 0$ such that $f(\boldsymbol{a}) \leq f(\boldsymbol{x})$ for all $\boldsymbol{x} \in B_r(\boldsymbol{a})$

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(iii) $f(\boldsymbol{a})$ is called a *local extremum* of f if and only if $f(\boldsymbol{a})$ is a local maximum or a local minimum of f.

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In fact the one-dimensional function

$$g(t) = f(a_1, \ldots, a_{j-1}, t, a_{j+1}, \ldots, a_n)$$

has a local extremum at $t = a_j$ for each $j = 1, \ldots, n$. Hence, by the one-dimensional theory

$$\frac{\partial f}{\partial x_j}(\boldsymbol{a}) = g'(a_j) = 0$$

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As in the one-dimensional case, $\nabla f(\boldsymbol{a}) = \mathbf{0}$ is necessary but not sufficient for $f(\boldsymbol{a})$ to be a local extremum.

Remark. There exist continuously differentiable functions that satisfy $\nabla f(\boldsymbol{a}) = \mathbf{0}$ such that $f(\boldsymbol{a})$ is neither a local maximum nor a local minimum.

Consider for $n = 2$

$$f(x, y) = y^2 - x^2$$

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It is easy to check that $\nabla f(\mathbf{0}) = \mathbf{0}$ but the origin is a saddle point see figure

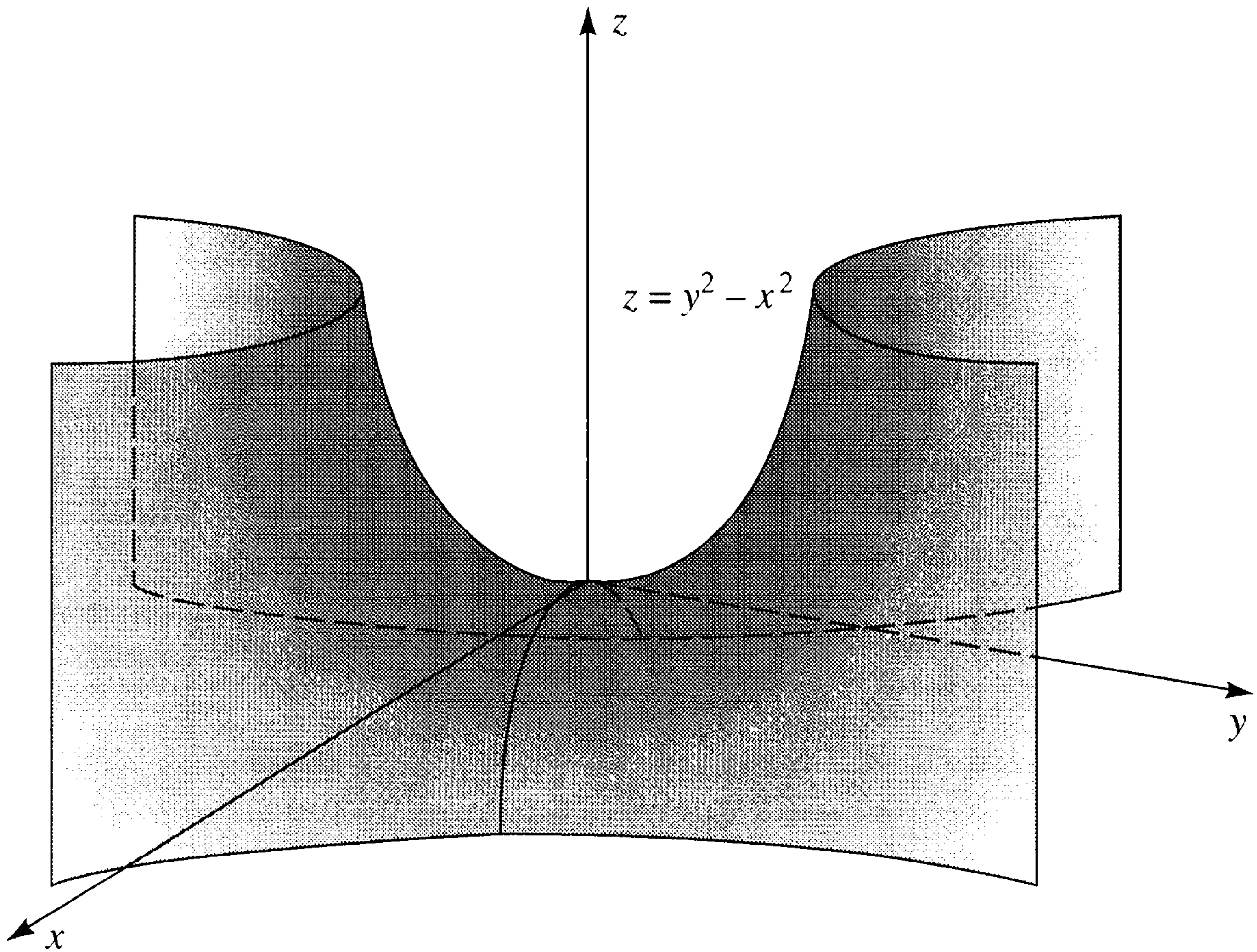

Figure 1: Saddle point

Definition. Let V be open in $\mathbb{R}^n$, let $\boldsymbol{a} \in V$, and let $f : V \to \mathbb{R}$ be differentiable at $\boldsymbol{a}$. Then $\boldsymbol{a}$ is called a *saddle point* of f if $\nabla f(\boldsymbol{a}) = \mathbf{0}$ and there is a $r_0 > 0$ such that given any $0 < \rho < r_0$ there are points $\boldsymbol{x}$, $\boldsymbol{y} \in B_\rho(\boldsymbol{a})$ that satisfy

$$f(\boldsymbol{x}) < f(\boldsymbol{a}) < f(\boldsymbol{y})$$

Hessian matrix. Let $V \subseteq \mathbb{R}^n$ an open set and let $f : V \to \mathbb{R}$ be a $\mathcal{C}^2$ function. The Hessian matrix of f at $\boldsymbol{x} \in V$ (or simply the Hessian) is the symmetric square matrix of second-order partial derivatives of f at $\boldsymbol{x}$:

$$H(f)(\boldsymbol{x}) := \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{x})\right]_{i,j=1,\ldots,n}$$

Test for extrema and saddle points

Theorem. Let V be open in $\mathbb{R}^2$, $(a,b) \in V$, and suppose that $f : V \to \mathbb{R}$ satisfies $\nabla f(a,b) = \mathbf{0}$. Suppose further that $f \in \mathcal{C}^2$ and set

$$D := f_{xx}(a,b) f_{yy}(a,b) - f_{xy}^2(a,b)$$

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Observe that

$$D = \det[H(f)(a,b)]$$

Examples.

$$\boldsymbol{f(x, y) = x^3 + 6xy - 3y^2 + 2}$$

has a saddle point in $(a, b) = (0, 0)$ and a local maximum in $(a, b) = (-2, -2)$

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$$\boldsymbol{f(x,y) = x^2 + y^3 - 2xy - y}$$

has a saddle point in $(a,b) = (-\frac{1}{3}, -\frac{1}{3})$ and a local minimum in $(a,b) = (1,1)$

Remark. In n variables a critical point $\boldsymbol{x}_0$ is a local minimum for $f \in \mathcal{C}^2$ if for each $k = 1, \ldots, n$

$$\det[H_k(f)(\boldsymbol{x}_0)] > 0$$

where $H_k(f)$ is the principal minor of order k of $H(f)$

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$$\det[(-1)^k H_k(f)(\boldsymbol{x}_0)] > 0$$

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Theorem (Lagrange multipliers) Let $m < n$, V be open in $\mathbb{R}^n$ and f, $g_j : V \to \mathbb{R}$ be $\mathcal{C}^1$ on V for $j = 1, 2, \ldots, m$. Suppose that rank of

$$\frac{\partial(g_1, \ldots g_m)}{\partial(x_1, \ldots, x_n)}$$

is m in $\boldsymbol{x}_0 \in V$ where $g_j(\boldsymbol{x}_0) = 0$ for $j = 1, 2, \ldots, m$ and suppose that $\boldsymbol{x}_0$ is a local extremum for f in the set

$$M = \{\boldsymbol{x} \in V : \ g_j(\boldsymbol{x}) = 0\}.$$

Then there exist scalars $\lambda_1, \ldots, \lambda_m$ such that

$$\nabla \left(f(\boldsymbol{x}_0) - \sum_{k=1}^{m} \lambda_k g(\boldsymbol{x}_0) \right) = \mathbf{0}$$

Sufficient conditions: Bordered Hessian

Find max or min of $f(x,y)$ under the constraint $g(x,y)=0$

Lagrangean $L(x,y,\lambda)=f(x,y)-\lambda g(x,y)$

After solving the system

$$\begin{cases} f_x'(x,y)-\lambda g_x'(x,y)=0, \\ f_y'(x,y)-\lambda g_y'(x,y)=0, \\ g(x,y)=0. \end{cases}$$

evaluate

$$\Lambda = \det \begin{bmatrix} L''_{xx} & L''_{xy} & g_x \\ L''_{xy} & L''_{yy} & g_y \\ g_x & g_y & 0 \end{bmatrix}$$

$\Lambda > 0$ maximum

$\Lambda < 0$ minimum

Cobb Douglas

$$f(x, y) = x^a y^{1-a} \to \max$$

$$\text{sub} \quad px + qy - c = 0$$

Assumptions $0 < a < 1$, p, q, $c > 0$.

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Assumptions $0 < a < 1$, p, q, $c > 0$.

Lagrangean $L(x,y;m) = f(x,y) - mw(x,y)$ where $w(x,y) = px + qy - c$. Critical point equations

$$L_x(x,y;m) = ax^{a-1}y^{1-a} - mp = 0 \tag{1a}$$

$$L_y(x,y;m) = (1-a)x^a y^{-a} - mq = 0 \tag{1b}$$

$$L_m(x,y;m) = px + qy - c = 0 \tag{1c}$$

Eliminating m between (1a) and (1b) we get

$$(1-a)px - aqy = 0 \tag{2a}$$

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Solving:

$$\begin{cases} x = \dfrac{ac}{p} \\ y = \dfrac{c(1-a)}{q} \\ m = (1-a)^{1-a} a^a p^{-a} q^{a-1} \end{cases}$$

The critical point is a maximum, in fact the bordered hessian is

$$\begin{bmatrix} (a-1)a\left(\frac{ac}{p}\right)^{a-2}\left(\frac{c-ac}{q}\right)^{1-a} & (1-a)a\left(\frac{ac}{p}\right)^{a-1}\left(\frac{c-ac}{q}\right)^{-a} & p \\ (1-a)a\left(\frac{ac}{p}\right)^{a-1}\left(\frac{c-ac}{q}\right)^{-a} & -\frac{a\left(\frac{ac}{p}\right)^{a}\left(\frac{c-ac}{q}\right)^{-a}q}{c} & q \\ p & q & 0 \end{bmatrix}$$

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so that

$$\det = \frac{a^{a-1}p^{2-a}q^{a+1}}{c(1-a)^a} > 0$$